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# Generalized stochastic equations

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**Abstract.** We consider the dynamic system of  $N$  particles describable by the fluctuation equations

$$\dot{x} = F(x, t)$$

where  $x$  and  $F$  are both  $N$ -dimensional vectors. The corresponding stochastic equations of the probability density  $\omega(X_m, t)$  for a subsystem of  $m \leq N$  particles are shown to be

$$\frac{\partial \omega(X_m, t)}{\partial t} = - \frac{\partial}{\partial x_j} \langle F_j(x, t) | X_m(t) = X_m \rangle \omega(X_m, t), \quad m = 1, 2, \dots, N.$$

Here the summation for the repeated index goes from 1 to  $m$  and  $X_m = (x_1, x_2, \dots, x_m)$ . For the Coulomb-potential case, this set is equivalent to the BBGKY hierarchy including the Liouville equation. Moment equations have also been obtained.

The set of exact stochastic equations and the set of Fokker-Planck equations are related in that the latter can be considered as a time expansion of the former on conditional averages. The derived Fokker-Planck equations differ from the conventional ones in that the coefficients are expressed as conditional averages. These extra conditions are very important for a class of problems in which the random force must be computed self-consistently.

## 1. Introduction

The Fokker-Planck equation has often been used to describe the distribution of a particle in a random field (Chandrasekhar 1943, Sturrock 1960, Stratonovich 1963). When such a technique is applied to a homogeneous plasma in order to obtain a kinetic equation, certain difficulties arise (Gasiorowicz *et al.* 1956). In the first place, the electric field must be determined consistently and the coefficients cannot be found explicitly without further assumptions. Moreover, the coefficients are given as averages of functionals of the random field (Sturrock 1966). A naive interpretation of these averages may give incorrect results. For example, in the derivation of the friction and diffusion coefficients of the Fokker-Planck equation in a plasma, Hubbard (1961) introduced the dielectrically screened field through physical arguments. It can be shown that the additional physical arguments are unnecessary if the set of generalized stochastic equations are used.

A set of generalized stochastic equations can be derived by a method similar to that used by Stratonovich (1963). In order to include the class of problems whose force must be computed self-consistently, previous forms of the Fokker-Planck equation must be modified. The modification comes about because of the possible correlation between the random force and the particle position. Details are given in § 2. Generalizations to conditional density functions, to higher dimensions and to the case of a more general fluctuation equation are discussed in §§ 3 and 4. In § 5 a set of exact stochastic equations is derived. An alternate derivation of the exact equations is shown in appendix 2. For Coulomb-potential problems this set of exact stochastic equations is shown in appendix 3 to be equivalent to the BBGKY hierarchy, including the Liouville equation. However, the new form given by (22) expresses clearly the importance of conditional averages. It also shows that the Fokker-Planck equation can be considered as time expansion on conditional averages. Moments of the exact stochastic equation are obtained in § 6.

## 2. Derivation of generalized stochastic equations

We are interested in the dynamic system describable by the fluctuation equation of the form

$$\dot{x} = \epsilon F(x, t) \tag{1}$$

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where the force  $F$  is a random function of  $x$  and  $t$  and the parameter  $\epsilon$  is used to indicate the smallness of the quantity. Stratonovich has shown that such an equation leads to a Fokker–Planck equation. Implicit in his assumption is that the force is statistically independent of the position of the particle. Such an assumption is invalid for the class of problems in which the force must be computed self-consistently. One such example is the plasma problem where the force is derived from particle Coulomb interactions. Therefore, Stratonovich’s result must be modified and generalized.

Let the increment of displacement be

$$z(t) = x(t) - x(0). \tag{2}$$

The probability density of  $x$  at  $t$  is then related to the probability density of  $z(t)$  by

$$\omega(x, t) = \int \omega(x - z, 0 | z(t) = z) \omega_z(z) dz \tag{3}^\dagger$$

where the integration limits go from  $-\infty$  to  $+\infty$ . The subscript on  $\omega$  denotes the process. Since  $x$  is the process of interest, it is omitted for simplicity. We note the condition  $z(t) = z$  on the first factor of the integrand. It comes about because the increment and the initial position may not be statistically independent. Expanding  $\omega(x - z, 0 | z(t) = z)$  by a Taylor series and carrying out the integration in (3) we obtain

$$\omega(x, t) = (1 + L)\omega(x, 0) \tag{4}$$

where the operator is given by

$$1 + L = \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{\partial}{\partial x} \right)^n \langle z^n(t) | x(0) = x \rangle. \tag{5}$$

An equation involving  $\omega$  alone can be obtained by taking partial derivatives with respect to  $t$  and by taking the inverse of (4). The result is

$$\partial \omega(x, t) / \partial t = \dot{L}(1 + L)^{-1} \omega. \tag{6}$$

Equation (6) is, formally, the same as that given by Stratonovich except that the moments are conditional moments as shown in (5). In order to express these moments explicitly in terms of  $F$  we expand  $z$  in a perturbation series

$$z = \epsilon z_1 + \epsilon^2 z_2 + \dots \tag{7}$$

† Consider the sum of two random processes

$$X = Y + Z. \tag{A}$$

In general  $Y$  and  $Z$  are not necessarily independent. The probability density function of  $x$  is given by

$$\omega_x(x) = \int \omega_Y(x - z | Z = z) \omega_Z(z) dz \tag{B}$$

where  $\omega_Y(x - z | Z = z)$  is the conditional probability density of  $Y = x - z$  given  $Z = z$  and  $\omega_Z(z)$  the probability density of  $z$ . For the processes related through (2) we can immediately obtain (3) by using formula (B). We note that in (3)  $\omega(x - z, 0 | z(t) = z)$  is a conditional probability density of  $x(t) = x - z$  at  $t = 0$  given  $z(t) = z$ .

In case there is another process  $W$  which is again not necessarily independent of  $Y$  and  $Z$ , equation (B) is still valid, provided we impose the extra condition  $W = w$  on all probability density functions. Generalization to higher conditional densities is obvious. These generalizations are needed in § 3.

Expanding  $F$  similarly in a Taylor series gives us

$$\epsilon F(x, t) = \epsilon F(x_0, t) + \epsilon \frac{\partial F(x_0, t)}{\partial x} (\epsilon z_1 + \epsilon^2 z_2 + \dots) + \dots \quad (8)$$

Substituting (7) and (8) into (1), we find

$$\begin{aligned} z_1(x_0, t) &= \int_0^t F(x_0, t_1) dt_1 \\ z_2(x_0, t) &= \int_0^t dt_2 \frac{\partial F(x_0, t_2)}{\partial x} \int_0^{t_2} F(x_0, t_1) dt_1 \\ &\dots \end{aligned} \quad (9)$$

The desired equation is obtained when (9) is substituted into (6), giving

$$\begin{aligned} \frac{\partial \omega(x, t)}{\partial t} &= \left[ -\epsilon \frac{\partial}{\partial x} \left\{ \langle F|x(0) = x \rangle + \epsilon \int_{-t}^0 \left\langle \frac{\partial F}{\partial x} F_\tau |x(0) = x \right\rangle d\tau \right\} \right. \\ &\quad + \epsilon^2 \frac{\partial^2}{\partial x^2} \int_{-t}^0 K[F, F_\tau |x(0) = x] d\tau \\ &\quad \left. + \epsilon^2 \frac{\partial}{\partial x} \left\{ \frac{\partial}{\partial x} \langle F|x(0) = x \rangle \int_{-t}^0 \langle F_\tau |x(0) = x \rangle d\tau \right\} \right] \omega(x, t) \\ &+ \dots \end{aligned} \quad (10)$$

Equation (10) is the generalized stochastic equation taking into account the correlation between the random force and the initial position of the particle. In equation (10),  $F = F(x, t)$ ,  $F_\tau = F(x, t + \tau)$ ,  $K(F_1, F_2) = \langle F_1 F_2 \rangle - \langle F_1 \rangle \langle F_2 \rangle$ , where  $\langle \rangle$  denotes the ensemble average. It should be noted that the starting point of the present derivation is (3) which is perfectly general for processes related by (2). The only requirement is that the process is sufficiently well behaved so that Taylor series exists for  $\omega(x - z, 0 | z(t) = z)$ . The fact that an equation of the form (10) applies to non-Markovian processes was pointed out by Bartlett (1955). The Markovian limit comes in when we let  $t$  be much larger than the correlation time, and cut off all terms higher than  $\epsilon^2$ . In this case the difference in the conditional average and the non-conditional average is of the order of  $\epsilon$  or higher. This means that all conditional averages in the third and fourth terms on the right-hand side of (10) can be approximated by the corresponding non-conditional ones. However, the conditional averages of the first and second terms on the right-hand side of (10) must be computed to the order  $\epsilon$ . This feature is important if the force depends on the unperturbed particle trajectories. When applied to the plasma case this gives rise to the screening effect.

### 3. Generalization to conditional density functions and to higher dimensions

It is often of interest to find the conditional density  $\omega(x, t | \bar{x}, \bar{t})$ , viz. the probability density of finding a particle at  $x$  at time  $t$  given the particle at  $\bar{x}$  at time  $\bar{t}$ . The extra condition  $\bar{x}(\bar{t}) = \bar{x}$  enters on all probability density functions in (3). Since the extra condition appears merely as a parameter, we can immediately conclude that  $\omega(x, t | \bar{x}, \bar{t})$  satisfies (10), except that we must impose the extra condition  $\bar{x}(\bar{t}) = \bar{x}$  on all averages. It is obvious that we may extend directly to the vector case and let

$$(\bar{x}, \bar{t}) = (\bar{x}_1, \bar{t}_1; \bar{x}_2, \bar{t}_2; \dots; \bar{x}_m, \bar{t}_m).$$

The results given here can also be generalized to higher dimensions. For a system of fluctuation equations

$$x_j = \epsilon F_j(x, t), \quad j = 1, 2, \dots, N \quad (11)$$

where  $x = (x_1, x_2, \dots, x_N)$ , the generalized stochastic equation can be shown to be (So 1967)

$$\begin{aligned} \frac{\partial \omega(x, t)}{\partial t} = & \left[ -\epsilon \frac{\partial}{\partial x_j} \left( \langle F_j \rangle_c + \epsilon \int_{-t}^0 \left\langle \frac{\partial F_j}{\partial x_k} F_{k\tau} \right\rangle_c d\tau \right) \right. \\ & + \epsilon^2 \frac{\partial^2}{\partial x_j \partial x_k} \int_{-t}^0 K(F_j, F_{k\tau})_c d\tau \\ & + \epsilon^2 \frac{\partial}{\partial x_j} \left( \frac{\partial}{\partial x_k} \langle F_j \rangle_c \right) \int_{-t}^0 \langle F_{k\tau} \rangle_c d\tau \left. \right] \omega(x, t) \\ & + \dots \end{aligned} \tag{12}$$

Here the subscript  $c$  indicates that the averages are subject to the condition  $x(0) = x$  and the subscript  $\tau$  indicates that the function is evaluated at  $t + \tau$ .

In a many-particle system equations of the form (12) can also be derived for the joint density functions giving rise to a hierarchy of generalized stochastic equations. For example, if we consider a subsystem of (11) with  $j$  running from 1 through  $m$ , the force in (11) is then a random function of  $(x_1, x_2, \dots, x_N)$ . The density function  $\omega(x_1, x_2, \dots, x_m, t)$  can be shown to satisfy (12) with summation running from 1 to  $m$ . The condition  $c$  now stands for  $x_1(0) = x_1, x_2(0) = x_2, \dots, x_m(0) = x_m$ . The discussion of this hierarchy of equations and especially their application to plasma problems will be considered in a later publication.

**4. Perturbation by small random forces**

The dynamic system given by (1) or (11) is rather restrictive. We wish to generalize it to the system describable by

$$\dot{x}_j = F_j^0 + \epsilon F_j \quad j = 1, 2, \dots, N. \tag{13}$$

The general method is to transform (13) into (11) where only small random forces are present. The transformation can be made if we know the unperturbed trajectories (or more generally, the integrals of the unperturbed system) as considered by So (1967). We shall illustrate the approach for the simple, but important, case where

$$\dot{x} = Ax + \epsilon F. \tag{14}$$

Here  $x$  and  $F$  are both  $N$ -dimensional vectors, and  $A = [a_{ij}]$  is an  $N \times N$  constant matrix. Making the transformation

$$y = e^{-At}x, \quad x = e^{At}y \tag{15}$$

we obtain

$$\dot{y} = \epsilon e^{-At}F. \tag{16}$$

If we let  $[\lambda_{ij}^{-1}(t)] = e^{-At}$ , then (16) is equivalent to

$$\dot{y}_i = \epsilon \lambda_{ij}^{-1}(t) \mathcal{F}_j(y, t) \quad i = 1, 2, \dots, N \tag{17}$$

where  $\mathcal{F}_j(y, t) = F_j(e^{At}y, t)$  and the convention of summation index is used.

A stochastic equation for  $\omega_Y(y, t)$ , the density of  $y(t)$ , may be immediately written down according to (12):

$$\begin{aligned} \frac{\partial \omega_Y(y, t)}{\partial t} = & \left\{ -\epsilon \frac{\partial}{\partial y_j} \left[ \langle \lambda_{ji}^{-1}(t) \mathcal{F}_i \rangle_{c'} + \epsilon \int_{-t}^0 \left\langle \lambda_{ji}^{-1} \frac{\partial \mathcal{F}_i}{\partial y_k} (\lambda_{kl}^{-1} \mathcal{F}_l)_\sigma \right\rangle_{c'} d\sigma \right] \right. \\ & + \epsilon^2 \frac{\partial^2}{\partial y_j \partial y_k} \int_{-t}^0 K[\lambda_{ji}^{-1} \mathcal{F}_i, (\lambda_{kl}^{-1} \mathcal{F}_l)_\sigma]_{c'} d\sigma \\ & \left. + \epsilon^2 \frac{\partial}{\partial y_j} \left[ \frac{\partial}{\partial y_k} \langle \lambda_{ji}^{-1} \mathcal{F}_i \rangle_{c'} \right] \int_{-t}^0 \langle (\lambda_{kl}^{-1} \mathcal{F}_l)_\sigma \rangle_{c'} d\sigma \right\} \omega_Y. \end{aligned} \tag{18}$$

Here the subscript  $c'$  refers to the condition  $y(0) = y$ . We wish now to transform (18)

back into the original variables. Details are given in appendix 1. For a conservative system,  $\omega_Y(e^{-At}x, t) = \omega(x, t)$  where  $\omega(x, t)$  is the density of  $x(t)$ . The resulting equation is

$$\begin{aligned} \frac{\partial \omega(x, t)}{\partial t} + a_{ij} x_j \frac{\partial \omega}{\partial x_i} &= -\epsilon \frac{\partial}{\partial x_j} \langle F_j \rangle_c \omega \\ &- \epsilon^2 \frac{\partial}{\partial x_k} \int_{-t}^0 d\sigma \lambda_{il}^{-1}(\sigma) \left\langle \frac{\partial F_k}{\partial x_i} F_l(e^{A\sigma} x, t + \sigma) \right\rangle_c \omega \\ &+ \epsilon^2 \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_j} \int_{-t}^0 d\sigma \lambda_{jl}^{-1}(\sigma) K[F_k, F_l(e^{A\sigma} x, t + \sigma)]_c \omega \\ &+ \epsilon^2 \frac{\partial}{\partial x_k} \left[ \frac{\partial}{\partial x_j} \langle F_k \rangle_c \right] \int_{-t}^0 d\sigma \lambda_{jl}^{-1}(\sigma) \langle F_l(e^{A\sigma} x, t + \sigma) \rangle_c \omega \end{aligned} \quad (19)$$

where the subscript  $c$  indicates that the averages are subject to the condition  $x(0) = e^{-At}x$ .

For the more general fluctuation equations (13) where the associated unperturbed system may not be autonomous, a stochastic equation similar to (18) can still be obtained. In such cases, it may be more convenient to work with  $\omega_Y$  than  $\omega$ .

In passing we remark that a hierarchy of generalized stochastic equations can also be obtained as that discussed in § 3.

## 5. Relations to the Liouville equation and the BBGKY hierarchy

In previous sections we have shown that fluctuation equations of the form (11) and (13) lead to a system of generalized stochastic equations. In applications these equations are truncated and hence are approximate. We note that in (12) and (18), when  $t \rightarrow 0$ , all higher-order terms vanish and therefore the resulting equation becomes exact.† For this limit, (12) reduces to (we set  $\epsilon = 1$  since the equation is exact)

$$\left. \frac{\partial \omega(x, t)}{\partial t} \right|_{t=0} = - \left. \frac{\partial \omega}{\partial x_j} \langle F_j(x, t) | x(0) = x \rangle \omega \right|_{t=0}. \quad (20)$$

Since we have chosen 0 arbitrarily as the initial time, in general we have

$$\frac{\partial \omega(x, t)}{\partial t} = - \frac{\partial}{\partial x_j} \langle F_j(x, t) | x(t) = x \rangle \omega. \quad (21)$$

For a subsystem of  $m \leq N$  particles, similar considerations give us the equation

$$\frac{\partial \omega(X_m, t)}{\partial t} = - \frac{\partial}{\partial x_j} \langle F_j(x, t) | X_m(t) = X_m \rangle \omega(X_m, t) \quad m = 1, 2, \dots, N \quad (22)$$

where the summation for the repeated index goes only from 1 to  $m$  and

$$X_m = (x_1, x_2, \dots, x_m), \quad X_m(t) = \{x_1(t), x_2(t), \dots, x_m(t)\}.$$

Equations (21) and (22) look deceptively simple. For the  $N$ -particle system with Coulomb force they lead directly to the BBGKY hierarchy and the Liouville equation. This is done in appendix 3. Therefore, the hierarchy given by (22) and the BBGKY hierarchy are just different modes of representing the same dynamic system. The generalized stochastic equations (12) and (19) can be considered as a time expansion of (21) on conditional averages. It is possible to obtain the kinetic equation for a plasma using the generalized stochastic equations instead of the BBGKY hierarchy. This will be done in the next paper.

† We assume that the series given by (12) and (18) are convergent. An alternate approach is given in appendix 2 which does not require such an assumption.

## 6. Moment equations

Consider the dynamic system described by

$$\left. \begin{aligned} \dot{\mathbf{x}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \mathbf{F}(\mathbf{x}, \mathbf{v}, t) \end{aligned} \right\} \quad (23)$$

The fundamental equation (21) now becomes

$$\frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot (\mathbf{v}f) + \frac{\partial}{\partial \mathbf{v}} \cdot \langle \mathbf{F} | \mathbf{x}(t) = \mathbf{x}, \mathbf{v}(t) = \mathbf{v} \rangle f = 0 \quad (24)$$

where we note  $\mathbf{x} = (x_1, x_2, x_3, v_1, v_2, v_3)$  and  $f$  is the one-particle density function. Note that (24) reduces to the Vlasov equation if  $\mathbf{F}$  is approximately independent of the conditions  $\mathbf{x}(t) = \mathbf{x}, \mathbf{v}(t) = \mathbf{v}$ . In general, such a reduction is not possible. For the Coulomb potential problem (24) is equivalent to the first equation in BBGKY hierarchy as shown in appendix 3.

Multiply (24) by 1,  $m\mathbf{v}$  and  $\frac{1}{2}m\mathbf{v}^2$  successively and integrate over velocity spaces; we obtain

$$\begin{aligned} \frac{\partial n}{\partial t} + \nabla \cdot (n \langle \mathbf{v} \rangle_v) &= 0 \\ \frac{\partial}{\partial t} (mn \langle \mathbf{v} \rangle_v) + \nabla \cdot (mn \langle \mathbf{v}\mathbf{v} \rangle_v) - mn \langle \langle \mathbf{F} | \mathbf{x}(t) = \mathbf{x}, \mathbf{v}(t) = \mathbf{v} \rangle \rangle_v &= 0 \\ \frac{\partial}{\partial t} (mn \langle \frac{1}{2}\mathbf{v}^2 \rangle_v) + \nabla \cdot (mn \langle \mathbf{v}\frac{1}{2}\mathbf{v}^2 \rangle_v) - mn \langle \langle \mathbf{v} \cdot \langle \mathbf{F} | \mathbf{x}(t) = \mathbf{x}, \mathbf{v}(t) = \mathbf{v} \rangle \rangle_v &= 0. \end{aligned} \quad (25)$$

The velocity average is denoted by

$$\langle \phi \rangle_v n = n_0 \int \phi f d^3v \quad (26)$$

where  $n_0$  is the average particle density, a constant, and  $n$  is the local probable number density. The moment equations (25) are exact but they are only formal since they are not closed. However, the form of (25) may be convenient for making certain suitable approximations.

## 7. Conclusion

A set of exact stochastic equations (22) has been derived for the dynamic system (11). The set is shown to be equivalent to the BBGKY hierarchy for the Coulomb potential problem. The moment equations are also obtained. The generalized stochastic equations (12) and (19) can be viewed as time expansion on conditional averages of (22). For times much larger than the correlation time these generalized stochastic equations can be truncated and lead directly to the Fokker-Planck equations. The derived Fokker-Planck equations differ from the conventional ones in that the coefficients are expressed as conditional averages. This new feature is important if the force on the particle depends on the particle trajectory as in the case of a gaseous plasma.

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## Appendix 1

We wish to indicate here intermediate steps to go from (18) to (19). We start with the

term on the left of (18). For a conservative system  $\det e^{-At} = 1$ ; we therefore have

$$\begin{aligned} \frac{\partial \omega_Y(y, t)}{\partial t} &= \frac{\partial \omega(x, t)}{\partial t} + \frac{\partial x_i}{\partial t} \frac{\partial \omega(x, t)}{\partial x_i} \\ &= \frac{\partial \omega(x, t)}{\partial t} + a_{ij} x_j \frac{\partial \omega(x, t)}{\partial x_i}. \end{aligned}$$

For the first term on the right of (18), we note that  $\lambda_{ji}^{-1}(t) \partial x_k / \partial y_j = \delta_{ik}$  and  $y(0) = x(0)$ . It can be evaluated straightforwardly as follows:

$$\begin{aligned} \frac{\partial}{\partial y_j} \langle \lambda_{ji}^{-1}(t) \mathcal{F}_i \rangle_c \omega_Y &= \frac{\partial x_k}{\partial y_j} \frac{\partial}{\partial x_k} \langle \lambda_{ji}^{-1}(t) F_i \rangle_c \omega \\ &= \frac{\partial}{\partial x_i} \langle F_i \rangle_c \omega. \end{aligned}$$

For the second term on the right of (18) we proceed as follows:

$$\begin{aligned} \frac{\partial}{\partial y_j} \int_{-t}^0 \langle \lambda_{ji}^{-1} \frac{\partial \mathcal{F}_i}{\partial y_k} (\lambda_{kl}^{-1} \mathcal{F}_l)_\sigma \rangle_c d\sigma \omega_Y \\ &= \frac{\partial}{\partial x_i} \int_{-t}^0 \langle \frac{\partial x_j}{\partial y_k} \frac{\partial F_i}{\partial x_j} (\lambda_{kl}^{-1}(t+\sigma) F_l [\exp\{A(t+\sigma)\} e^{-At} x, t+\sigma]) \rangle_c d\sigma \omega \\ &= \frac{\partial}{\partial x_i} \int_{-t}^0 \langle \delta_{jk} \frac{\partial F_i}{\partial x_j} \lambda_{kl}^{-1}(\sigma) F_l (e^{A\sigma} x, t+\sigma) \rangle_c d\sigma \omega \\ &= \frac{\partial}{\partial x_i} \int_{-t}^0 \lambda_{ji}^{-1}(\sigma) \langle \frac{\partial F_i}{\partial x_j} F_l (e^{A\sigma} x, t+\sigma) \rangle_c d\sigma \omega \end{aligned}$$

where we have made use of the relation  $\lambda_{kl}^{-1}(t+\sigma) = \lambda_{kh}^{-1}(t) \lambda_{hl}^{-1}(\sigma)$ . The remaining terms in (18) can be evaluated similarly.

### Appendix 2

Here we wish to prove the generalization of (21). Since we shall not make the restrictive assumption (14) and the derivation is exact, we go back to (11) with  $\epsilon = 1$ . Let  $\psi_j(t, t_0, x_0) = x_j(t)$  be the solution of (11) through the initial point  $(t_0, x_0)$ . Here  $x_0 = (x_{10}, x_{20}, \dots, x_{N0})$ . Let  $J$  be the Jacobian defined by  $J = \partial(\psi_1, \psi_2, \dots, \psi_N) / \partial(x_{10}, x_{20}, \dots, x_{N0})$ . For convenience we take  $J > 0$ . Then the initial density and the density at  $t$  are related through

$$\omega(x_0, 0) = \omega(\psi(t, t_0, x_0), t) J.$$

Since the left-hand side is not a function of  $t$ , its time derivative vanishes. The time derivative of the right side yields

$$J \left[ \frac{\partial \omega(\psi, t)}{\partial t} + \sum_{j=1}^N F_j(\psi, t) \frac{\partial \omega(\psi, t)}{\partial x_j} \right] + \omega(\psi, t) \frac{dJ}{dt} = 0.$$

Now along the solution

$$\frac{dJ}{dt} = J \sum_{j=1}^N \frac{\partial F_j}{\partial x_j}$$

(Kaplan 1958). Consequently the above equation reduces to

$$\frac{\partial \omega(x, t)}{\partial t} + \sum_{j=1}^N \frac{\partial}{\partial x_j} F_j(x, t) \omega(x, t) = 0. \tag{A1}$$



It is a simple matter to prove that the equation given above is equivalent to the Liouville equation. Integrating over  $x_{m+1}, \dots, x_N$ , we obtain

$$\frac{\partial \omega(X_m, t)}{\partial t} + \sum_{j=1}^m \frac{\partial}{\partial x_j} \langle F_j | X_m(t) = X_m \rangle \omega(X_m, t) = 0 \quad m = 1, 2, \dots, N \quad (\text{A2})$$

which is identical with (22).

### Appendix 3

We wish to show that the stochastic equations (22) lead directly to the BBGKY hierarchy for the Coulomb potential problem. In this case  $x_1 = (\mathbf{x}_1, \mathbf{v}_1)$  and we let

$$f_1 = f(\mathbf{x}_1, \mathbf{v}_1, t) = V\omega(x_1, t), \quad f_2 = f(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, t) = V^2\omega(x_1, x_2, t).$$

Equation (22) becomes

$$\frac{\partial f_1}{\partial t} + \frac{\partial}{\partial \mathbf{x}_1} \cdot (\mathbf{v}_1 f_1) + \frac{\partial}{\partial \mathbf{v}_1} \cdot \langle \mathbf{F}_1 | \mathbf{x}_1(t) = \mathbf{x}_1, \mathbf{v}_1(t) = \mathbf{v}_1 \rangle f_1 = 0. \quad (\text{A3})$$

The conditional average of Coulomb force on electron 1 is given by

$$\begin{aligned} \langle \mathbf{F}_1 | \mathbf{x}_1(t) = \mathbf{x}_1, \mathbf{v}_1(t) = \mathbf{v}_1 \rangle &= \left\langle -\frac{\partial \phi_1}{m \partial \mathbf{x}_1} \middle| \mathbf{x}_1(t) = \mathbf{x}_1, \mathbf{v}_1(t) = \mathbf{v}_1 \right\rangle \\ &= \sum_{j=2}^N -\frac{1}{m} \int \frac{\partial \phi(\mathbf{x}_1 - \mathbf{x}_j)}{\partial \mathbf{x}_1} \frac{f_2/V^2}{f_1/V} d^3x_j d^3v_j \\ &= -\frac{N-1}{mVf_1} \int \frac{\partial \phi(\mathbf{x}_1 - \mathbf{x}_2)}{\partial \mathbf{x}_1} f_2(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, t) d^3x_2 d^3v_2 \quad (\text{A4}) \end{aligned}$$

where  $\phi(\mathbf{x}_1 - \mathbf{x}_j) = e^2/|\mathbf{x}_1 - \mathbf{x}_j|$ . Substituting (A3) into (A5), we obtain

$$\frac{\partial f_1}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial f_1}{\partial \mathbf{x}_1} - \frac{N-1}{m} \frac{\partial}{\partial \mathbf{v}_1} \cdot \int \frac{\partial \phi(\mathbf{x}_1 - \mathbf{x}_2)}{\partial \mathbf{x}_1} f_2(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, t) d^3x_2 d^3v_2 = 0 \quad (\text{A5})$$

which is the first equation in the BBGKY hierarchy. The entire hierarchy including the Liouville equation can be obtained in a similar manner.

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